Stochastic calculus of semi-Dirichlet forms with application to complement value problem for non-local operators

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Outline

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$$
\begin{aligned}
Lu &= \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right)+\sum_{i=1}^{d}b_i(x)\frac{\partial u}{\partial x_i}+(c(x)+\operatorname{div}\hat{b}(x))u \\
&\quad +\mathrm{PV}\int (u(y)-u(x))k(x,y)dy.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}(u,v) &= (-Lu,v) \\
&= \frac{1}{2}\sum_{i,j=1}^{d}\int a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx-\sum_{i=1}^{d}\int b_i(x)\frac{\partial u}{\partial x_i}v(x)dx \\
&\quad -\int c(x)u(x)v(x)dx+\sum_{i=1}^{d}\int \hat{b}_i(x)\frac{\partial (uv)}{\partial x_i}dx \\
&\quad -\int\int (u(y)-u(x))v(x)k(x,y)dydx.
\end{aligned}
$$

Lower bounded semi-Dirichlet form

E: locally compact separable metric space.

m: positive Radon measure on E with full support.

$D(\mathcal{E})$: dense subspace of $L^2(E;m)$.

$\mathcal{E}(\cdot,\cdot)$: bilinear form defined on $D(\mathcal{E}) \times D(\mathcal{E})$.

- (lower boundedness) There exists $\beta > 0$ such that

$$\mathcal{E}_\beta(u,u) := \mathcal{E}(u,u) + \beta(u,u) \geq 0,\ \forall u \in D(\mathcal{E}).$$

- (weak sector condition) There exists $K \geq 1$ such that

$$|\mathcal{E}(u,v)| \leq \sqrt{\mathcal{E}_\beta(u,u)}\sqrt{\mathcal{E}_\beta(v,v)},\ \forall u,v \in D(\mathcal{E}).$$

- (closedness) $D(\mathcal{E})$ is closed w.r.t. the norm $\sqrt{\mathcal{E}_\alpha(\cdot,\cdot)}$, $\alpha > \beta$.

- (Markovian condition)

$$(0\vee u)\wedge 1 \in D(\mathcal{E}) \text{ and } \mathcal{E}((0\vee u)\wedge 1, u-(0\vee u)\wedge 1) \geq 0,\ \forall u \in D(\mathcal{E}).$$

$(\mathcal{E}, D(\mathcal{E}))$ is regular if $D(\mathcal{E}) \cap C_c(E)$ is uniformly dense in $C_c(E)$ and is $\mathcal{E}_\alpha$-dense in $D(\mathcal{E})$ for $\alpha > \beta$.

Any regular lower bounded semi-Dirichlet form is associated with a Hunt process.

Use the theory of Markov processes to study

$$\begin{aligned} Lu &= \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} + (c(x) + \operatorname{div}\hat{b}(x))u \\ &\quad + \mathrm{PV}\int (u(y) - u(x))k(x,y)dy. \end{aligned}$$

Fukushima's decomposition for Dirichlet forms

Suppose $(\mathcal{E}, D(\mathcal{E}))$ is a regular Dirichlet form on $L^2(E;m)$ with associated Hunt process $(X_t)_{t\geq 0}$. Let $u \in D(\mathcal{E})$. Then

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^u + N_t^u,\ t \geq 0.$$

M_t^u: martingale additive functional of $(X_t)_{t\geq 0}$.

N_t^u: continuous additive functional with $e(N^u) = 0$.

For additive functional $(A_t)_{t\geq 0}$ of $(X_t)_{t\geq 0}$,

$$e(A) := \lim_{t\downarrow 0} \frac{1}{2t} E_m(A_t^2).$$

Fukushima type decomposition for semi-Dirichlet forms
Z.M. Ma, Sun and L.F. Wang (2016)

Suppose $(\mathcal{E}, D(\mathcal{E}))$ is a regular semi-Dirichlet form on $L^2(E;m)$ with associated Hunt process $(X_t)_{t\geq 0}$.

ζ: lifetime of X.

ζ_i: the totally inaccessible part of ζ.

$I(\zeta) := [[0,\zeta[[\bigcup[[\zeta_i]]$. $I(\zeta)$ is a predictable set of interval type.

Let $u \in D(\mathcal{E})_{loc}$. Then the following two assertions are equivalent to each other.

(i) There exist unique $M^u \in \mathcal{M}_{loc}^{I(\zeta)}$ and AF N^u locally of zero quadratic variation such that

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^u + N_t^u,\ t \geq 0.$$

(ii)

$$\mu_u(dx) := \int_E (\tilde{u}(x) - \tilde{u}(y))^2 J(dy, dx) \text{ is a smooth measure.}$$

Fukushima type decomposition for semi-Dirichlet forms
C.Z. Chen, L. Ma and Sun (2018)

Let $u \in D(\mathcal{E})_{loc}$. Define

$$F_t^u := \sum_{0<s\leq t} (\tilde{u}(X_s) - \tilde{u}(X_{s-}))1_{\{|\tilde{u}(X_s)-\tilde{u}(X_{s-})|>1\}}.$$

Then there exist unique $Y^u \in \mathcal{M}_{loc}^{I(\zeta)}$ and AF Z^u locally of zero quadratic variation such that

$$\tilde{u}(X_t) - \tilde{u}(X_0) = Y_t^u + Z_t^u + F_t^u, \ t \geq 0.$$

Extended Nakao integral

Denote $A_t^u = \tilde{u}(X_t) - \tilde{u}(X_0)$.

Let $\Phi \in C^2(\mathbb{R}^n)$ and $u_1, \ldots, u_n \in D(\mathcal{E})_{loc}$. Then,

$$A_t^{\Phi(u)} = \sum_{i=1}^{n} \int_0^t \Phi_i(\tilde{u}(X_{s-}))dA_s^{u_i} + \frac{1}{2}\sum_{i,j=1}^{n} \int_0^t \Phi_{ij}(\tilde{u}(X_s))d\langle M^{u_i,c}, M^{u_j,c}\rangle_s$$

$$+ \sum_{0<s\leq t} \left[\Delta\Phi(\tilde{u}(X_s)) - \sum_{i=1}^{n} \Phi_i(\tilde{u}(X_{s-}))\Delta u_i(X_s)\right]$$

on $[0, \zeta)$ P_x-$a.s.$ for $\mathcal{E}$-$q.e.$ $x \in E$, where

$$\Phi_i(x) = \frac{\partial \Phi}{\partial x_i}(x), \quad \Phi_{ij}(x) = \frac{\partial^2 \Phi}{\partial x_i \partial x_j}(x), \quad i, j = 1, \ldots, n,$$

and $u = (u_1, \ldots, u_n)$.

Outline

Classical Dirichlet problem:

$$\begin{cases} \Delta u = 0 & \text{in } \ D, \\ u = g & \text{on } \ \partial D. \end{cases}$$

D: bounded domain of $\mathbb{R}^d$, ∂D: boundary of D.

g: real-valued continuous function on ∂D.

Kakutani (1944):

(X_t, P_x): Brownian motion on $\mathbb{R}^d$.

$\tau = \inf\{t > 0 : X_t \notin D\}$: first exit time from D.

$$u(x) = E_x[g(X_\tau)].$$

Dirichlet boundary value problem for second order elliptic differential operators:

$$\begin{cases} Lu = 0 & \text{in } D, \\ u = g & \text{on } \partial D. \end{cases}$$

with

$$Lu = \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} + (c(x) + \operatorname{div}\hat{b}(x))u.$$

Complement value problem for non-local operators:

$$\begin{cases} Lu = f & \text{in } D, \\ u = g & \text{on } D^c. \end{cases}$$

with

$$Lu = \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} + (c(x) + \operatorname{div}\hat{b}(x))u$$
$$+\mathrm{PV}\int_{\mathbb{R}^d}(u(y) - u(x))k(x,y)dy.$$

Applications in peridynamics, particle systems with long range interactions, fluid dynamics, image processing, etc.

semigroup approach: Bony, Courrège and Priouret

classical PDE approach: Garroni and Menaldi

viscosity solution approach: Barles, Chasseigne and Imbert; Arapostathisa, Biswasb and Caffarelli

Hilbert space approach: Hoh and Jocob; Felsinger, Kassmann and Voigt

Recent results on interior and boundary regularity of solutions.

D: bounded Lipschitz domain of $\mathbb{R}^d$.

We consider the complement value problem:

$$\begin{cases} (\Delta + a^{\alpha}\Delta^{\alpha/2} + b\cdot\nabla + c)u + f = 0 & \text{in } D, \\ u = g & \text{on } D^c. \end{cases}$$

Fractional Laplacian operator $\Delta^{\alpha/2}$:

$$\Delta^{\alpha/2}\phi(x) = \lim_{\varepsilon\to 0}\mathcal{A}(d,-\alpha)\int_{\{|x-y|\geq\varepsilon\}}\frac{\phi(y)-\phi(x)}{|x-y|^{d+\alpha}}dy, \quad \phi\in C_c^{\infty}(\mathbb{R}^d).$$

$$L := \Delta + a^\alpha \Delta^{\alpha/2} + b \cdot \nabla.$$

Z.Q. Chen and Hu (2015): the martingale problem for $(L, C_c^\infty(\mathbb{R}^d))$ is well-posed for every initial value $x \in \mathbb{R}^d$.

$$q_\rho(t,z) = t^{-d/2} \exp\left(-\frac{\rho|z|^2}{t}\right) + t^{-d/2} \wedge \frac{t}{|z|^{d+\alpha}}, \quad t > 0, z \in \mathbb{R}^d.$$

The Markov process X associated with L has a jointly continuous transition density function $p(t,x,y)$:

$$C_1 q_{C_2}(t, x-y) \le p(t,x,y) \le C_3 q_{C_4}(t,x-y), \quad (t,x,y) \in (0,T] \times \mathbb{R}^d \times \mathbb{R}^d.$$

$$e(t) := e^{\int_0^t c(X_s)ds}, \quad t \geq 0.$$

Theorem (Sun, 2018a): There exists $M > 0$ such that if $\|c^+\|_{L^{p\vee 1}} \leq M$, then for any $f \in L^{2(p\vee 1)}(D; dx)$ and $g \in B_b(D^c)$, there exists a unique $u \in B_b(\mathbb{R}^d)$ satisfying $u|_D \in H^{1,2}_{loc}(D) \cap C(D)$ and

$$\begin{cases} (\Delta + a^\alpha \Delta^{\alpha/2} + b \cdot \nabla + c)u + f = 0 & \text{in } D, \\ u = g & \text{on } D^c. \end{cases}$$

Moreover, u has the expression

$$u(x) = E_x\left[e(\tau)g(X_\tau) + \int_0^\tau e(s)f(X_s)ds\right], \quad x \in \mathbb{R}^d.$$

In addition, if g is continuous at $z \in \partial D$ then

$$\lim_{x\to z} u(x) = u(z).$$

$$H^{1,2}_{loc}(D) := \{u : u\phi \in H^{1,2}_0(D) \text{ for any } \phi \in C_c^\infty(D)\}.$$

$(\Delta + a^\alpha \Delta^{\alpha/2} + b \cdot \nabla + c)u + f = 0$ is understood in the distribution sense: for any $\phi \in C_c^\infty(D)$,

$$\int_D \langle \nabla u, \nabla \phi \rangle dx + \frac{a^\alpha \mathcal{A}(d, -\alpha)}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{d+\alpha}} dxdy$$
$$- \int_D \langle b, \nabla u \rangle \phi dx - \int_D cu\phi dx - \int_D f\phi dx = 0.$$

Corollary: If $c \leq 0$, then for any $f \in L^{2(p\vee 1)}(D; dx)$ and $g \in B_b(D^c)$ satisfying g is continuous on ∂D, there exists a unique $u \in B_b(\mathbb{R}^d)$ such that u is continuous on $\overline{D}$, $u|_D \in H^{1,2}_{loc}(D)$, and

$$\begin{cases} (\Delta + a^\alpha \Delta^{\alpha/2} + b\cdot\nabla + c)u + f = 0 \ \text{ in } D, \\ u = g \ \text{ on } D^c. \end{cases}$$

Moreover, u has the expression

$$u(x) = E_x\left[e(\tau)g(X_\tau) + \int_0^\tau e(s)f(X_s)ds\right], \quad x \in \mathbb{R}^d.$$

$$\begin{cases} \mathcal{E}^0(\phi,\psi) = \int_{\mathbb{R}^d}\langle\nabla\phi,\nabla\psi\rangle dx + \frac{a^\alpha\mathcal{A}(d,-\alpha)}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(\phi(x)-\phi(y))(\psi(x)-\psi(y))}{|x-y|^{d+\alpha}}dxdy \\ \qquad\qquad - \int_{\mathbb{R}^d}\langle b,\nabla\phi\rangle\psi dx, \quad \phi,\psi\in D(\mathcal{E}^0), \\ D(\mathcal{E}^0) = H^{1,2}(\mathbb{R}^d). \end{cases}$$

Then, $(\mathcal{E}^0, H^{1,2}(\mathbb{R}^d))$ is a regular lower-bounded semi-Dirichlet form on $L^2(\mathbb{R}^d; dx)$.

$$(X_t, P_x) \Leftrightarrow L = \Delta + a^\alpha\Delta^{\alpha/2} + b\cdot\nabla \Leftrightarrow (\mathcal{E}^0, H^{1,2}(\mathbb{R}^d)).$$

$$u(x)=E_x\left[e(\tau)g(X_\tau)+\int_0^\tau e(s)f(X_s)ds\right],\ \ x\in\mathbb{R}^d.$$

U: open set of $\mathbb{R}^d$.

$$G_0^U(x,y)\le\begin{cases}\frac{C}{|x-y|^{d-2}}, & d\ge 3,\\ C\ln\left(1+\frac{1}{|x-y|}\right), & d=2,\\ C, & d=1.\end{cases}$$

Lemma: There exists $C>0$ such that

$$\sup_{x\in D}E_x\left[\int_0^\tau v(X_s)ds\right]\le C\|v\|_{L^{p\vee 1}},\ \ \forall v\in L_+^{p\vee 1}(D).$$

By Khasminskii's inequality, there exists $C > 0$ such that for any $v \in L_+^{p\vee 1}(D)$ satisfying $\|v\|_{L^{p\vee 1}} \leq C$, we have

$$\sup_{x\in D} E_x \left[e^{\int_0^\tau v(X_s)ds} \right] < \infty,$$

and

$$E_x \left[\left| \int_0^\tau e^{\int_0^s v(X_t)dt} f(X_s)ds \right| \right] \leq C \left(E_x \left[e^{\int_0^\tau 4v(X_s)ds} \right] \right)^{1/4} \left(E_x \left[\tau^2 \right] \right)^{1/4} \|f^2\|_{L^{p\vee 1}}^{1/2}.$$

Then, there exists $M > 0$ such that if $\|c^+\|_{L^{p\vee 1}} \leq M$, then for any $f \in L^{2(p\vee 1)}(D; dx)$ and $g \in B_b(D^c)$, $u \in B_b(\mathbb{R}^d)$.

$$\begin{aligned}
u(x) &= E_x\left[e(\tau)g(X_\tau) + \int_0^\tau e(s)f(X_s)ds\right] \\
&= E_x[u(X_t)] + E_x[-u(X_t)1_{\{\tau\le t\}} + (e(t)-1)u(X_t)1_{\{\tau>t\}} \\
&\quad +e(\tau)g(X_\tau)1_{\{\tau\le t\}} + \int_0^{t\wedge\tau} e(s)f(X_s)ds] \\
&:= u_t(x) + \sum_{i=1}^4 \varepsilon_t^{(i)}.
\end{aligned}$$

u_t is continuous in D and each $\varepsilon_t^{(i)}$ converges to 0 uniformly on any compact subset of D as $t\to 0$. Hence u is continuous in D.

$$
\begin{aligned}
u(x) &= E_x\left[e(\tau)g(X_\tau) + \int_0^\tau e(s)f(X_s)ds\right] \\
&= E_x[g(X_\tau)] + E_x\left[w(X_t)1_{\{\tau>t\}} + \int_0^{t\wedge\tau}(f+cu)(X_s)ds\right] \\
&:= \xi(x) + w(x).
\end{aligned}
$$

Suppose that $\overline{D} \subset B(0,N)$ for some $N \in \mathbb{N}$. Denote $\Omega = B(0,N)$.

Lemma: Let $\gamma \geq 0$. For any compact set K of Ω, there exist $\delta > 0$ and $\vartheta_1, \vartheta_2 \in (0,\infty)$ such that for any $x, y \in K$ satisfying $|x-y| < \delta$, we have

$$\begin{cases} \frac{\vartheta_1}{|x-y|^{d-2}} \leq G_\gamma^\Omega(x,y) \leq \frac{\vartheta_2}{|x-y|^{d-2}}, & \text{if } d \geq 3, \\ \vartheta_1 \ln \frac{1}{|x-y|} \leq G_\gamma^\Omega(x,y) \leq \vartheta_2 \ln \frac{1}{|x-y|}, & \text{if } d = 2. \end{cases}$$

Let $z \in \partial D$. z is a regular point of D and D^c for the Brownian motion in $\mathbb{R}^d$. Therefore, z is a regular point of D and D^c for (X_t, P_x) by Kanda (1967).

If g is continuous at $z \in \partial D$, then $\lim_{x \to z} \xi(x) = \xi(z)$.

Lemma: Let U be an open set of $\mathbb{R}^d$. Suppose that φ is a measurable function on $\mathbb{R}^d$ which belongs to the Kato class. Then, we have

$$\lim_{t\to 0}\sup_{x\in U} E_x\left[\int_0^t |\varphi(X_s^U)|ds\right] = 0.$$

Lemma: For any $t > 0$ and $z \in \partial D$, we have

$$\lim_{\substack{x\to z\\ x\in D}}\left(\sup_{y\in D} p^D(t,x,y)\right) = 0.$$

Therefore,

$$\lim_{t\to 0}\sup_{x\in D}\left|E_x\left[w(X_t)1_{\{\tau>t\}} + \int_0^{t\wedge\tau}(f+cu)(X_s)ds\right]\right| = 0.$$

$$
\begin{aligned}
u(x) &= E_x\left[e(\tau)g(X_\tau) + \int_0^\tau e(s)f(X_s)ds\right] \\
&= E_x[g(X_\tau)] + E_x\left[w(X_t)1_{\{\tau>t\}} + \int_0^{t\wedge\tau}(f+cu)(X_s)ds\right] \\
:&= \xi(x) + w(x).
\end{aligned}
$$

$\xi \in H^{1,2}_{loc}(D)$ and $\mathcal{E}^0(\xi,\phi) = 0$, $\forall \phi \in C_c^\infty(D)$.

$w \in H_0^{1,2}(D)$ and $\mathcal{E}^0(w,\phi) = (f+cu,\phi)$, $\forall \phi \in C_c^\infty(D)$.

$\Rightarrow u = \xi + w \in H^{1,2}_{loc}(D)$ and $\mathcal{E}^0(u,\phi) = \mathcal{E}^0(\xi+w,\phi) = (f+cu,\phi)$.

There exists $\beta_0 > 0$ such that

$$\int_{\mathbb{R}^d} |b|^2 \phi^2 dx \leq \frac{1}{2} \int_{\mathbb{R}^d} |\nabla \phi|^2 dx + \beta_0 \int_{\mathbb{R}^d} |\phi|^2 dx, \ \ \forall \phi \in H^{1,2}(\mathbb{R}^d).$$

Define

$$\mathcal{E}^0_\beta(\phi, \psi) = \mathcal{E}^0(\phi, \psi) + \beta(\phi, \psi), \ \ \phi, \psi \in H^{1,2}(\mathbb{R}^d).$$

Then, $(\mathcal{E}^0_\beta, H^{1,2}(\mathbb{R}^d))$ is a regular semi-Dirichlet form on $L^2(\mathbb{R}^d; dx)$ for any $\beta > \beta_0$.

Let $\{D_n\}_{n\in\mathbb{N}}$ be a sequence of relatively compact open subsets of D such that $\overline{D}_n \subset D_{n+1}$ and $D = \cup_{n=1}^\infty D_n$, and $\{\chi_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $C_c^\infty(D)$ such that $0 \leq \chi_n \leq 1$ and $\chi_n|_{D_n} = 1$.

$$
\begin{aligned}
& \mathcal{E}^0(\xi\chi_n, \phi) \\
= \ & \lim_{t\to 0}\frac{1}{t}\int_{D_m}\phi(x)E_x[e^{-\beta(t\wedge\tau_{D_m})}\xi(X_{t\wedge\tau_{D_m}})(1-\chi_n(X_{t\wedge\tau_{D_m}}))]dx \\
= \ & \lim_{t\to 0}\frac{1}{t}\int_{D_m}\phi(x)E_x[1_{\{\tau_{D_m}\le t\}}e^{-\beta\tau_{D_m}}\xi(X_{t\wedge\tau_{D_m}})(1-\chi_n(X_{t\wedge\tau_{D_m}}))]dx \\
\le \ & \lim_{t\to 0}\frac{1}{t}\int_{D_m}|\phi(x)|\left[\int_0^t\int_{\overline{D}_m^c}\left(\xi(y)(1-\chi_n(y))\int_{D_m}\frac{p^{D_m}(s,x,z)}{|z-y|^{d+\alpha}}dz\right)dyds\right]dx \\
= \ & \lim_{t\to 0}\frac{1}{t}\int_{D_m}|\phi(x)|\left[\int_0^t\int_{D\cap D_n^c}\left(\xi(y)(1-\chi_n(y))\int_{D_m}\frac{p^{D_m}(s,x,z)}{|z-y|^{d+\alpha}}dz\right)dyds\right]dx \\
\le \ & \|\phi\|_\infty\|\xi\|_\infty\delta^{-(d+\alpha)}|D||D\cap D_n^c|.
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{E}^0(\xi\chi_n,\phi)\\
= &\int_{\mathbb{R}^d}\langle\nabla(\xi\chi_n),\nabla\phi\rangle dx-\int_{\mathbb{R}^d}\langle b,\nabla(\xi\chi_n)\rangle\phi dx\\
&+\frac{a^\alpha\mathcal{A}(d,-\alpha)}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{((\xi\chi_n)(x)-(\xi\chi_n)(y))(\phi(x)-\phi(y))}{|x-y|^{d+\alpha}}dxdy\\
= &\mathcal{E}^0(\xi,\chi_n\phi)-\int_{\mathbb{R}^d}(L\chi_n)\xi\phi dx-2\int_{\mathbb{R}^d}\langle\nabla\xi,\nabla\chi_n\rangle\phi dx\\
&-a^\alpha\mathcal{A}(d,-\alpha)\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}\frac{(\xi(y)-\xi(x))(\chi_n(y)-\chi_n(x))}{|x-y|^{d+\alpha}}dy\right]\phi(x)dx\\
= &\mathcal{E}^0(\xi,\phi)+a^\alpha\mathcal{A}(d,-\alpha)\int_{D_m}\int_{D\cap D_n^c}\frac{\xi(y)(1-\chi_n(y))}{|x-y|^{d+\alpha}}dy\phi(x)dx.
\end{aligned}
$$

We will show that there exists $M > 0$ such that if $\|c^+\|_{L^{p\vee 1}} \leq M$, then $v \equiv 0$ is the unique function in $B_b(\mathbb{R}^d)$ satisfying $v|_D \in H^{1,2}_{loc}(D) \cap C(D)$ and

$$\begin{cases} \mathcal{E}^0(v, \phi) = (cv, \phi), \ \ \forall \phi \in C_c^\infty(D), \\ v = 0 \ \ \text{on } D^c. \end{cases}$$

$$
\begin{aligned}
& \mathcal{E}_\beta^0(v\chi_n, \phi) \\
= \; & \int_{\mathbb{R}^d} \langle \nabla(v\chi_n), \nabla\phi\rangle dx - \int_{\mathbb{R}^d} \langle b, \nabla(v\chi_n)\rangle \phi dx + (\beta, v\chi_n\phi) \\
& + \frac{a^\alpha \mathcal{A}(d,-\alpha)}{2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{((v\chi_n)(x) - (v\chi_n)(y))(\phi(x) - \phi(y))}{|x-y|^{d+\alpha}} dxdy \\
= \; & \mathcal{E}^0(v, \chi_n\phi) - \int_{\mathbb{R}^d} (L\chi_n) v\phi dx - 2\int_{\mathbb{R}^d} \langle \nabla v, \nabla \chi_n\rangle \phi dx + (\beta, v\chi_n\phi) \\
& - a^\alpha \mathcal{A}(d,-\alpha) \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \frac{(v(y) - v(x))(\chi_n(y) - \chi_n(x))}{|x-y|^{d+\alpha}} dy\right] \phi(x) dx \\
= \; & \Big((c+\beta)v\chi_n - (L\chi_n)v - 2\langle \nabla v, \nabla \chi_n\rangle \\
& - a^\alpha \mathcal{A}(d,-\alpha) \int_{\mathbb{R}^d} \frac{(v(y) - v(\cdot))(\chi_n(y) - \chi_n(\cdot))}{|\cdot - y|^{d+\alpha}} dy, \phi\Big) \\
:= \; & (\theta_n, \phi).
\end{aligned}
$$

For $n > m$, define

$$\eta_{m,n}(x) = E_x^\beta[(v\chi_n)(X_{\tau_{D_m}})], \quad x \in \mathbb{R}^d.$$

We have $\eta_{m,n}(x) = E_x^\beta[\eta_{m,n}(X_{t\wedge\tau_{D_m}})], t \geq 0, \ x \in D_m$, $\eta_{m,n}(x) = v\chi_n(x)$ for q.e.-$x \in D_m^c$, and

$$\mathcal{E}_\beta^0(v\chi_n, \phi) = \mathcal{E}_\beta^0(v\chi_n - \eta_{m,n}, \phi), \quad \forall \phi \in H_0^{1,2}(D_m).$$

For dx-a.e. $x \in D_m$,

$$(v\chi_n - \eta_{m,n})(x) = E_x^\beta[(v\chi_n - \eta_{m,n})(X_{t\wedge\tau_{D_m}})] + E_x^\beta\left[\int_0^{t\wedge\tau_{D_m}} \theta_n(X_s)ds\right], \ \ \forall t \geq 0.$$

For dx-a.e. $x \in D$,

$$v(x) = E_x^\beta[v(X_t)1_{\{\tau>t\}}] + E_x^\beta\left[\int_0^{t\wedge\tau}((c+\beta)v)(X_s)ds\right], \ \ \forall t \geq 0.$$

Define

$$\mathcal{I}_t = v(X_t)1_{\{\tau>t\}} + \int_0^{t\wedge\tau}((c+\beta)v)(X_s)ds.$$

Then, $(\mathcal{I}_t)_{t\geq 0}$ is a martingale under P_x^β for dx-a.e. $x \in D$.

By integration by parts formula for semi-martingales, we can show that

$$v(x) = E_x[e(t)v(X_t)1_{\{\tau>t\}}], \quad dx - \text{a.e. } x \in D.$$

D: bounded Lipschitz domain of $\mathbb{R}^d$.

We consider the complement value problem:

$$\begin{cases} (\Delta + a^\alpha \Delta^{\alpha/2} + b \cdot \nabla + c + \mathrm{div}\hat{b})u + f = 0 & \text{in } D, \\ u = g & \text{on } D^c. \end{cases}$$

$\hat{b} = (\hat{b}_1, \dots, \hat{b}_d)^*$ satisfying $|\hat{b}| \in L^{2(p\vee 1)}(D; dx)$, $c + \mathrm{div}\hat{b} \le h$ for some $h \in L_+^{p\vee 1}(D; dx)$ in the distribution sense.

Let $\phi \in H^{1,2}(\mathbb{R}^d)$. By Ma, Sun and Wang (2016), ϕ admits a unique Fukushima type decomposition

$$\tilde{\phi}(X_t) - \tilde{\phi}(X_0) = M_t^{\phi} + N_t^{\phi}, \quad t \geq 0,$$

where $(M_t^{\phi})_{t\geq 0}$ is a locally square integrable martingale additive functional and $(N_t^{\phi})_{t\geq 0}$ is a continuous additive functional locally of zero quadratic variation.

By the Lax-Milgram theorem, for any vector field $\eta \in L^2(\mathbb{R}^d; dx)$, there exists a unique $\eta^H \in H^{1,2}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \langle \eta, \nabla \phi \rangle dx = \mathcal{E}^0_\gamma(\eta^H, \phi), \quad \forall \phi \in H^{1,2}(\mathbb{R}^d),$$

where $\gamma = \beta_0 + 1$.

For $\phi \in C_c^\infty(\mathbb{R}^d)$, we have

$$\int_0^t \mathrm{div}\phi(X_s) ds = N_t^{\phi^H} - \gamma \int_0^t \phi^H(X_s) ds, \quad t \geq 0.$$

$$e(t) := e^{\int_0^t c(X_s)ds + N_t^{\hat{b}^H} - \gamma \int_0^t \hat{b}^H(X_s)ds}, \quad t \geq 0.$$

Theorem (Sun, 2018b): There exists $M > 0$ such that if $\|h\|_{L^{p\vee 1}} \leq M$, then for any $f \in L^{4(p\vee 1)}(D; dx)$ and $g \in B_b(D^c)$, there exists a unique $u \in B_b(\mathbb{R}^d)$ satisfying $u|_D \in H^{1,2}_{loc}(D) \cap C(D)$ and

$$\begin{cases} (\Delta + a^\alpha \Delta^{\alpha/2} + b \cdot \nabla + c + \mathrm{div}\hat{b})u + f = 0 & \text{in } D, \\ u = g & \text{on } D^c. \end{cases}$$

Moreover, u has the expression

$$u(x) = E_x \left[e(\tau)g(X_\tau) + \int_0^\tau e(s)f(X_s)ds \right] \quad \text{for q.e. } x \in D.$$

In addition, if g is continuous at $z \in \partial D$ then

$$\lim_{x \to z} u(x) = u(z).$$

Corollary: If $c+\mathrm{div}\hat{b}\le 0$, then for any $f\in L^{4(p\vee 1)}(D;dx)$ and $g\in B_b(D^c)$ satisfying g is continuous on ∂D, there exists a unique $u\in B_b(\mathbb{R}^d)$ such that u is continuous on $\overline{D}$, $u|_D\in H^{1,2}_{loc}(D)$, and

$$\begin{cases} (\Delta + a^{\alpha}\Delta^{\alpha/2} + b\cdot\nabla + c + \mathrm{div}\hat{b})u + f = 0 & \text{in } D,\\ u = g & \text{on } D^c. \end{cases}$$

Moreover, u has the expression

$$u(x) = E_x\left[e(\tau)g(X_\tau) + \int_0^\tau e(s)f(X_s)ds\right] \quad \text{for q.e. } x\in D.$$

Define

$$J(x)=\frac{1_{\{|x|<1\}}e^{-\frac{1}{1-|x|^2}}}{\int_{\{|y|<1\}}e^{-\frac{1}{1-|y|^2}}dy},\quad x\in\mathbb{R}^d.$$

For $k\in\mathbb{N}$ and $x\in\mathbb{R}^d$, set $J_k(x)=k^dJ(kx)$ and

$$\hat{b}_k=\hat{b}*J_k,\quad c_k=c*J_k,\quad h_k=h*J_k.$$

Define

$$e_k(t) := e^{\int_0^t (c_k + \mathrm{div}\hat{b}_k)(X_s)ds}, \quad t \geq 0.$$

The unique bounded continuous weak solution to the complement value problem

$$\begin{cases} (\Delta + a^\alpha \Delta^{\alpha/2} + b \cdot \nabla + c_k + \mathrm{div}\hat{b}_k)u_k + f = 0 \ \text{ in } D \\ u_k = g \ \text{ on } D^c \end{cases}$$

is given by

$$u_k(x) = E_x\left[e_k(\tau)g(X_\tau) + \int_0^\tau e_k(s)f(X_s)ds\right], \quad x \in \mathbb{R}^d.$$

Lemma: Suppose that $u \in B_b(\mathbb{R}^d)$ satisfying $u|_D \in H^{1,2}_{loc}(D)$ and $(\Delta + a^\alpha \Delta^{\alpha/2} + b \cdot \nabla + c + \mathrm{div}\hat{b})u + f = 0$ in D. Then, $u|_D$ has a locally Hölder continuous version.

Weak Harnack inequality:

$\mathcal{L} := \Delta + a^\alpha \Delta^{\alpha/2} + b \cdot \nabla + c + \mathrm{div}\hat{b}$.

There exist positive constants c_1, c_2 and ϱ_0 such that for any $x_0 \in \mathbb{R}^d$, $R \in (0,1)$, $v \in H^{1,2}(B_R(x_0))$ satisfying $v \geq 0$ in $B_R(x_0)$ and $(-\mathcal{L}v, \phi) \geq 0$ for any nonnegative $\phi \in C_c^\infty(B_R(x_0))$, we have

$$\begin{aligned}\inf_{B_{R/4}(x_0)} v \quad &\geq \quad c_1 \left(|B_{R/2}(x_0)|^{-1} \int_{B_{R/2}(x_0)} v^{\varrho_0} dx \right)^{1/\varrho_0} \\ &\quad - c_2 R^2 \sup_{x \in B_{R/2}(x_0)} \int_{\mathbb{R}^d \setminus B_R(x_0)} \frac{v^-(z)}{|x-z|^{d+\alpha}} dz.\end{aligned}$$

Lemma: Let C, R be two positive constants and μ be a function on $\mathbb{R}^d$ with $\mathrm{supp}[\mu] \subset B_R(0)$.

(i) Suppose $d \geq 2$. Then, there exist positive constants C_1, C_2 which are independent of μ such that for any $t > 0$ and $x \in D$,

$$\int_0^t \int_{y\in\mathbb{R}^d} (s^{-d/2}\exp(-\frac{C|x-y|^2}{s}) + s^{-d/2} \wedge \frac{s}{|x-y|^{d+\alpha}})|\mu(y)|dyds$$
$$\leq C_1(t^{\beta+1-d/2} + t^{\delta})(\int_{y\in\mathbb{R}^d} |\mu(y)|^p dy)^{1/p}$$

and

$$\int_0^t \int_{y\in\mathbb{R}^d} (s^{-(d+1)/2}\exp(-\frac{C|x-y|^2}{s}) + s^{-(d+1)/2} \wedge \frac{s}{|x-y|^{d+1+\alpha}})|\mu(y)|dyds$$
$$\leq C_2(t^{(1-\gamma)/2} + t^{\delta})(\int_{y\in\mathbb{R}^d} |\mu(y)|^{2p} dy)^{1/(2p)}.$$

(ii) Suppose $d = 1$. Then, for any $t > 0$ and $x \in D$,

$$\begin{aligned} &\int_0^t \int_{-\infty}^{\infty} (s^{-1/2} \exp(-\frac{C|x-y|^2}{s}) + s^{-1/2} \wedge \frac{s}{|x-y|^{1+\alpha}})|\mu(y)|dyds \\ \leq \quad & 4t^{1/2} \int_{-\infty}^{\infty} |\mu(y)|dy, \end{aligned}$$

and there exists a positive constant C_3 which is independent of μ such that for any $t > 0$ and $x \in D$,

$$\begin{aligned} &\int_0^t \int_{-\infty}^{\infty} (s^{-1} \exp(-\frac{C|x-y|^2}{s}) + s^{-1} \wedge \frac{s}{|x-y|^{2+\alpha}})|\mu(y)|dyds \\ \leq \quad & C_3(t^{1/6} + t^{\delta})(\int_{-\infty}^{\infty} |\mu(y)|^2 dy)^{1/2}. \end{aligned}$$

Uniqueness of solution

We will show that $v \equiv 0$ is the unique function in $B_b(\mathbb{R}^d)$ satisfying $v|_D \in H^{1,2}_{loc}(D) \cap C(D)$ and

$$\begin{cases} (\Delta + a^\alpha \Delta^{\alpha/2} + b \cdot \nabla + c + \mathrm{div}\hat{b})v = 0 \ \ \text{in } D, \\ v = 0 \ \ \text{on } D^c. \end{cases}$$

$$\mathcal{E}_\beta^0(v\chi_n,\phi)=(\theta_n,\phi)-\int_{\mathbb{R}^d}\langle\hat{b},\nabla(v\chi_n\phi)\rangle dx.$$

For dx-a.e. $x\in E_l\cap D$,

$$\begin{aligned} v(x) &= E_x^\beta[v(X_{t\wedge\tau_{E_l\cap D}})]+E_x^\beta\left[\int_0^{t\wedge\tau_{E_l\cap D}}((c+\beta)v)(X_s)ds\right] \\ &\quad +E_x^\beta\left[\int_0^{t\wedge\tau_{E_l\cap D}}v(X_{s-})dN_s^{\beta,\hat{b}^H}\right],\quad \forall t\geq 0. \end{aligned}$$